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Complex Chebyshev Approximation with the Local Haar Condition

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Let X be a compact space and $C(X)$ the space of continuous complex valued functions on X. For $g \in C(X)$ define

$$\| g \| = \sup\{| g(x)| : x \in X\}.$$

Let F be a nonlinear approximating function with parameter vector $A = [a_1, ..., a_n]$ taken from P, a subset of complex n-space. We say that A is *locally best* to f if for all B $(\neq A)$ in a neighborhood N of A,

$$\|f - F(A, .)\| \leqslant \|f - F(B, .)\|.$$

If the inequality is strict, we say A is *strongly locally best*. In this note we characterize locally best approximation by parameters at which F satisfies the local Haar condition defined below.

Preliminaries

We define a norm for parameter vectors,

$$\| A \| = \max\{| a_k | : k = 1, ..., n\}.$$

Denote the partial derivative of F with respect to the kth parameter component a_k by F_k. Define

$$D(A, B, x) = \sum_{k=1}^{n} F_k(A, x)\, b_k, \qquad B = (b_1, ..., b_n).$$

DEFINITION. F has the *local Haar property* at $A \in P$ on $W \subset X$ if

(i) the space

$$S(A) = \{D(A, B, .): B \in C_n\}$$

is a Haar subspace of dimension n on W.

(ii) there is a neighborhood N of A such that $N \subset P$ and $F(B, .) \in C(X)$ for $B \in N$.

(iii) Define

$$R(A, B, x) := F(A + B, x) - F(A, x) - D(A, B, x),$$

then

$$\| R(A, B, .)\| = o(\| B \|) \qquad \text{as} \qquad \| B \| \to 0.$$

We do not assume that F has the local Haar property at all of its parameters in P.

We do not assume that $S(A)$ is a Haar subspace on X. One reason is that we may wish to consider approximation problems in which all approximants vanish on a closed set V. Another reason is that Haar subspaces of dimension >1 can exist only on quite restricted X [3]; in particular, the author knows of no case where X is larger than the extended complex plane.

Hypothesis (iii) is satisfied if F has a first order Taylor expansion in its parameters with second order remainder.

Characterization of Locally Best Approximation

Let us fix f and let $M(A)$ be the set of points of X such that

$$| f(x) - F(A, x)| = \| f - F(A, .)\|.$$

If $F(A, .)$ is continuous, $| f - F(A, .)|$ is continuous and $M(A)$ is a closed nonempty set.

THEOREM 1. *Let F have the local Haar property at A on $M(A)$. A necessary and sufficient condition for A to be locally best to $f \not\equiv F(A, .)$ on X is that for all $B \not\equiv 0$,*

$$\eta(B) = \inf\{\mathrm{Re}[\overline{(f(x) - F(A, x))} \cdot D(A, B, x)] : x \in M(A)\} < 0. \qquad (1)$$

Proof. Sufficiency. Suppose A is not strongly locally best; then there is a sequence $\{B_k\}$, $\| B_k \| \to 0$, such that

$$\| f - F(A + B_k , .)\| \leqslant \| f - F(A, .)\|.$$

We, therefore, have

$$\mathrm{Re}[\overline{(f(x) - F(A, x))} \cdot (F(A + B_k , x) - F(A, x))] \geqslant 0, \qquad x \in M(A),$$

and by (iii),

$$\mathrm{Re}[\overline{(f(x) - F(A, x))} \cdot (D(A , B_k , x) + R(A, B_k , x))] \geqslant 0, \qquad x \in M(A). \quad (2)$$

Define $B_k' = B_k/\| B_k \|$; then $\{B_k'\}$ has an accumulation point B, $\| B \| = 1$. Assume without loss of generality that $\{B_k'\} \to B$. Let

$$\mu = \inf\{\mathrm{Re}[\overline{(f(x) - F(A, x))} \cdot D(A, B, x)]: x \in M(A)\}.$$

Suppose $\mu < 0$. There is a point x of $M(A)$ such that

$$\mathrm{Re}[\overline{(f(x) - F(A, x))} \cdot D(A, B, x)] = \mu < 0. \tag{3}$$

Consider the sequence

$$\mathrm{Re}[\overline{(f(x) - F(A, x))} \cdot (D(A, B_k, x) + R(A, B_k, x))]/\| B_k \|.$$

As $D(A, B_k, x)/\| B_k \| = D(A, B_k', x)$ and $R(A, B_k, x) = o(\| B_k \|)$, this sequence tends to the left-hand side of (3). This contradicts (2) and so $\mu \geqslant 0$. Hence (1) does not hold.

Necessity. Define for $B \in C_n$,

$$\phi(B, x) = \mathrm{Re}[\overline{(f(x) - F(A, x))}\, D(A, B, x)].$$

Suppose there is $B \neq 0$ with $\eta(B) \geqslant 0$. First consider the case with $\eta(B) = 0$. Let $H = \phi(B, x)$. Let Z be the set of zeros of $D(A, B, .)$ on $M(A)$. By (i), Z has no more than $n - 1$ points. Condition (i) guarantees that there exists C such that

$$\phi(C, x) > 0, \qquad x \in Z.$$

There is an open subset U of $M(A)$ containing Z such that

$$\phi(C, x) > 0, \qquad x \in U. \tag{4}$$

If $U = M(A)$, $\eta(C) > 0$. We now suppose that $V = M(A) \sim U$ is non-empty. It is a closed subset of a compact set. Define $\sigma = \inf\{H(x): x \in V\}$.

By continuity of H and compactness of V, there is an $x \in V$ such that $H(x) = \sigma$. But $H(x) \geqslant 0$ for $x \in M(A)$ and H vanishes only on Z, so $\sigma > 0$. Select λ such that

$$\lambda\phi(C, x) = \phi(\lambda C, x) > -\sigma. \tag{5}$$

From (1) and (5) we have

$$\phi(B + \lambda C, x) > 0, \qquad x \in U. \tag{6}$$

From (5) and the definition of σ,

$$\phi(B + \lambda C, x) > \phi(B, x) + \phi(\lambda C, x) > \sigma + (-\sigma) = 0, \qquad x \in V. \tag{7}$$

Combining (6) and (7), we have

$$\phi(B + \lambda C, x) > 0, \qquad x \in U \cup V = X.$$

Thus $\eta(B + \lambda C) > 0$. We need only, therefore, consider the case where $\eta(B) > 0$. It is known in this case that A is not locally best [2, Theorem 8, p. 306].

A consequence of the sufficiency argument is

COROLLARY. *Let A be locally best to f and F have the local Haar property at A on $M(A)$; then A is strongly locally best.*

REMARK. If $S(A)$ is not of dimension n, (1) may not be sufficient for local optimality. An example is given in the later section on exponential-polynomial products.

An Irreducible Set for Local Optimality

By arguments similar to those of [1, Section 3] (but using the version of the theorem of Caratheodory dealing with complex numbers, from whence comes $2n + 1$) we obtain

THEOREM 2. *Let A be locally best to f on X and F have the local Haar property at A on a set containing $M(A)$. There is a subset T of X (containing at least $n + 1$ points and not more than $2n + 1$ points) such that*

$$|f(x) - F(A, x)| = \|f - F(A, .)\|, \qquad x \in T,$$

A is strongly locally best to f on T, and no smaller such subset exists.

Local Existence of Locally Best Approximations

THEOREM. *Let X be a metric space. Let A be locally best to f in neighborhood N of A and F have the local Haar property on $M(A)$. There exists $\epsilon > 0$ such that if $\|f - g\| < \epsilon$, g has a locally best approximation in N.*

Proof. By the local Haar property there exists a neighborhood N' of A such that $N' \subset P$ and $F(B, .) \in C(X)$ for $B \in N'$. By the corollary to Theorem 1,

$$\|f - F(A, .)\| < \|f - F(B, .)\|, \qquad B \in N \sim A.$$

Let L be a closed neighborhood of A in $N \cap N'$. It follows that there exists $\epsilon > 0$ such that

$$\|f - F(A, .)\| < \|f - F(B, .)\| - 2\epsilon, \qquad B \in \text{Bndry}(L).$$

Let $\|f - g\| < \epsilon$; then

$$\|f - F(A, .)\| < \|g - F(B, .)\|, \qquad B \in \text{Bndry}(L). \tag{8}$$

As $\| g - F(C, .)\|$ depends continuously on $C \in L$ and L is closed, there is $C^* \in L$ such that

$$\| g - F(C^*, .)\| = \inf\{\| g - F(C, .)\|: C \in L\}.$$

By (8), $C^* \in \mathrm{Int}(L)$ and

$$\| g - F(C^*, .)\| \leqslant \| g - F(C, .)\|, \qquad C \in \mathrm{Int}(L).$$

Rational Approximation

Let X be a compact subset of the complex plane containing the origin and at least $l + m$ other points,

$$F(A, x) = P(A, x)/Q(A, x) = \sum_{k=0}^{l} a_k x^k \Big/ \left(1 + \sum_{k=1}^{m} a_{l+k} x^k\right),$$

and let the parameter space be

$$\mathscr{P} = \{A: Q(A, x) \neq 0 \text{ for } x \in X\}.$$

There is no loss of generality in having the constant term in the denominator equal to one; if it were zero, $F(A, .)$ would have a pole at zero or would have common factors in numerator and denominator. Define $\bar{Q}(B, x) = Q(B, x) - 1$. We have

$$D(A, B, x) = P(B, x)/Q(A, x) - P(A, x)\, \bar{Q}(B, x)/Q^2(A, x),$$

$$E(A, B, x) = Q^2(A, x)\, D(A, B, x) = P(B, x)\, Q(A, x) - P(A, x)\, \bar{Q}(B, x).$$

$T(A) = \{E(A, ., .)\}$ is a linear space of dimension at most $l + m + 1$. An approximant $F(A, .)$ is said to be nondegenerate if it cannot be written as $P(B, x)/\bar{Q}(B, x)$. It is readily seen that $F(A, .)$ is nondegenerate if $P(A, .)/Q(A, .)$ have no common factors and $a_l a_{l+m} \neq 0$. Suppose $F(A, .)$ is nondegenerate. The only zero element of $T(A)$ is $E(A, 0, .)$; hence $T(A)$ is a linear space of dimension $l + m + 1$. Suppose $E(A, B, .)$ has $l + m + 1$ zeros on X. It is a power polynomial of degree at most $l + m$ and so must be identically zero. Hence $T(A)$ is a Haar subspace of dimension $l + m + 1$ on X. We conclude that the local Haar condition is satisfied at A on X if $F(A, .)$ is nondegenerate. This remains true if we let the parameter space $\mathscr{P}'$ be any open subset of $\mathscr{P}$. In particular this is true if the parameter space is

$$\mathscr{P}_+ = \{A: \mathrm{Re}\, Q(A, x) > 0 \text{ for } x \in X\}.$$

Polynomial-Rational Sums

Let X be a compact subset of the complex plane. Define

$$F(A, x) = \sum_{k=1}^{l} a_k/(1 + a_{l+k}x) + \sum_{k=1}^{m} a_{2l+k}x^{k-1}.$$

We will consider only parameters A such that

$$1 + a_{l+k}x \neq 0 \text{ for all } x \in X, k = 1,..., l. \tag{9}$$

Also in the case $m > 0$, there is no point in permitting any one of $a_{n+1} ,..., a_{2n}$ to vanish since any constant part can be put in through choice of a_{2l+1}. We therefore require

$$a_{l+k} \neq 0, k = 1,..., l \text{ if } m > 0. \tag{10}$$

Parameters A satisfying (9), (10) will be called *admissible*. An admissible parameter is called *standard* if $a_{l+1} ,..., a_{2l}$ are distinct. Any admissible parameter has an equivalent standard parameter. A standard parameter A is called *nondegenerate* if $a_1 ,..., a_l$ are nonzero. If A is nondegenerate, the corresponding $S(A)$ is a Haar subspace of dimension $2l + m$ (see the proof of Lemma 2 of [5]) and F has the local Haar property at A on X.

The case where X is a compact subset of the extended complex plane containing ∞, in which case only $m = 0$ and $m = 1$ are of interest, are handled using the approach just given and the approach of [5].

Exponential Polynomial Products

Let X be a compact subset of the complex plane and define for $A = (a_0 ,..., a_n)$,

$$P(A, x) = \sum_{k=1}^{n} a_k x^{k-1}, \qquad F(A, x) = \exp(a_0 x) P(A, x).$$

We have

$$D(A, B, x) = \exp(a_0 x)[b_0 x P(A, x) + P(B, x)].$$

If $a_n \neq 0$, $P(A, .)$ is a polynomial of exact degree $n - 1$, $xP(A, x)$ is of exact degree n, and $[b_0 x P(A, x) + P(B, x)]$ is the space of polynomials of degree n. Hence $\{D(A, B, .): B \in C_{n+1}\}$ is a Haar subspace of dimension $n + 1$ on X. Our theory applies to $F(A, .)$ with $a_n \neq 0$. If $a_n = 0$ our theory may not apply.

EXAMPLE. Let $n = 2$ and approximate $f(x) = 1 - x^2$ on $X = \{-1, 0, 1\}$. Consider $F(A, .) = \frac{1}{2}$, which has $a_n = 0$. $f - F(A, .)$ attains $-\frac{1}{2}$, $\frac{1}{2}$, $-\frac{1}{2}$, respectively, on X. It can be shown by arguments used in linear Chebyshev approximation that (1) holds, yet $F(A, .)$ is not locally best even in the real version of the problem [4].

Approximation by ϕ-Polynomials

Let X be a compact subset of complex space. Let ϕ be a function and Z be an open subset of the complex plane containing the origin on which ϕ is differentiable. There exists an open subset Y of the complex plane such that $a_2 x \in Z$ for $a_2 \in Y$, $x \in X$. Define

$$F(A, x) = a_1 \phi(a_2 x), \qquad P = \{(a_1, a_2): a_2 \in Y\}.$$

We have

$$D(A, B, x) = b_1 \phi(a_2 x) + b_2 a_1 x \phi'(a_2 x).$$

If $a_1 = 0$, $S(A)$ is of dimension 1. If $a_1 \neq 0$ and $a_2 = 0$, $S(A)$ is a Haar subspace of dimension 2 if and only if $\phi(0)\, \phi'(0) \neq 0$. If $a_1 a_2 \neq 0$, it is seen from the determinant definition of Haar subspace that $S(A)$ is a Haar subspace of dimension 2 if $x\phi'(x)/\phi(x)$ is $1:1$ on Z. We might have to restrict Z to guarantee this.

Transformations

Let s be a continuous complex function on X and $V = \{x: s(x) = 0\}$. Let ϕ be a continuous mapping from the complex plane into the extended complex plane. Define $G(A, x) = s(x)\, \phi(F(A, x))$ for $A \in P$. Let the parameter space of G be $P_1 = \{A: G(A, x) \in C(X), A \in P\}$.

THEOREM 3. *Let F have the local Haar property at A on W and $A \in P_1$. Let ϕ have a continuous nonvanishing derivative on $y = \{y: F(A, x) = y$ for some $x \in X\}$, and for $y_0 \in Y$, y in an open set containing y, let*

$$\phi(y - y_0) = \phi(y_0) + \phi'(y_0)(y - y_0) + o(y - y_0).$$

Then G has the Haar property at A on $W \sim V$ (if this has n points).

To prove this we argue as in the corresponding result in [1, p. 752].

The theorem can be applied to approximation by transformations of *linear* families satisfying the Haar condition.

The case where $s > 0$ corresponds to the introduction of a positive weight function. In the following, $*$ denotes multiplication.

COROLLARY. *Let F have the local Haar property at $A \in P$ on X, a compact subset of complex space containing at least $n + 1$ points. Then $x * F$ has the local Haar property at $A \in P$ on $X \sim \{0\}$.*

This applies if F is the approximating function for ratios of power polynomials described earlier. $x * F$ is a natural approximating form for a function f vanishing at zero.

COROLLARY. *Let F have the local Haar property at $A \in P$ on W; then* $\exp(F)$ *has the local Haar property at $A \in P$ on W.*

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